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On the affine self-similarities of the three-dimensional Penrose pattern

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Abstract. We prove that the vertex set $\mathcal{Q}$ of the usual three-dimensional Penrose tiling admits an infinite number of independent scaling factors and an infinite number of inflation centres. More exactly, we prove that there exist an infinite number of real numbers α and an infinite number of points $q \in \mathcal{Q}$ such that $r \in \mathcal{Q} \implies \alpha(r-q)+q \in \mathcal{Q}$.

A subset $\mathcal{P}$ of the usual three-dimensional Euclidean space $\mathbb{E}_3 = (\mathbb{R}^3, \langle , \rangle)$ is said [7, 1] to *be invariant under the affine similarity* of centre r_0 and scaling factor $\alpha \neq 0$

$$A : \mathbb{E}_3 \quad\longrightarrow\quad \mathbb{E}_3 : r \quad\mapsto\quad Ar = \alpha(r-r_0)+r_0 \tag{1}$$

if $r \in \mathcal{P} \Longrightarrow Ar \in \mathcal{P}$. In this case, A is called a *self-similarity* of $\mathcal{P}$.

The three-dimensional Penrose tiling, its strip projection construction [5, 6], and the invariance of the corresponding vertex set $\mathcal{Q}$ under the similarity $\mathbb{E}_3 \longrightarrow \mathbb{E}_3 : r \mapsto \tau^3 r$, where $\tau = \frac{1}{2}(1+\sqrt{5})$ are well known. The review of these results in a revisited mathematical formalism [3, 4, 8, 2] presented in the first part of the paper will allow us to prove the existence of a one-dimensional quasicrystal $\mathcal{C}$ such that $\mathcal{Q}$ is invariant under the similarity $r \mapsto \alpha r$, for any $\alpha \in \mathcal{C} - \{0\}$. In the case where $\alpha \in \mathcal{C}$ satisfies an additional condition we prove the existence of an infinite set $\mathcal{Q}_\alpha \subset \mathcal{Q}$ such that $\mathcal{Q}$ is invariant under the affine similarity $r \mapsto \alpha(r-q)+q$, for any $q \in \mathcal{Q}_\alpha$.

Starting from the set of all the vertices of a regular icosahedron, for example, $\mathcal{I} = \{e_1, e_2, \ldots, e_6, -e_1, -e_2, \ldots, -e_6\}$, where

$$\begin{array}{lll} e_1 = (1, 0, \tau) & e_3 = (\tau, 1, 0) & e_5 = (-1, 0, \tau) \\ e_2 = (\tau, -1, 0) & e_4 = (0, \tau, 1) & e_6 = (0, -\tau, 1) \end{array} \tag{2}$$

one considers the orthogonal decomposition $\mathbb{E}_6 = \mathbb{E}_6^{\parallel} \oplus \mathbb{E}_6^{\perp}$, where $\mathbb{E}_6^{\parallel}$ is the subspace generated by the vectors

$$e^1 = (1, \tau, \tau, 0, -1, 0) \qquad e^2 = (0, -1, 1, \tau, 0, -\tau) \qquad e^3 = (\tau, 0, 0, 1, \tau, 1). \tag{3}$$

The group of all the isometries of the space $\mathbb{E}_3$ which leave the set $\mathcal{I}$ invariant is isomorphic to the icosahedral group

$$Y = 235 = \left\langle a,\ b \,\middle|\, a^5 = b^2 = (ab)^3 = e \right\rangle \tag{4}$$

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and is generated by the transformations

$$a(x,y,z)=\left(\frac{1}{2}x-\frac{\tau}{2}y+\frac{\tau-1}{2}z,\ \frac{\tau}{2}x+\frac{\tau-1}{2}y-\frac{1}{2}z,\ \frac{\tau-1}{2}x+\frac{1}{2}y+\frac{\tau}{2}z\right)$$
$$b(x,y,z)=(-x,-y,z). \tag{5}$$

It can be regarded as an orthogonal irreducible representation of this group. The Y-invariant set $\mathcal{I}$ coincides with the orbit $Y(1,0,\tau)=\{g(1,0,\tau)\mid g\in Y\}$ of this representation, and the generators of Y define the signed permutations

$$a=\begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ e_1 & e_3 & e_4 & e_5 & e_6 & e_2 \end{pmatrix} \qquad b=\begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ e_5 & -e_2 & -e_3 & e_6 & e_1 & e_4 \end{pmatrix}. \tag{6}$$

The corresponding transformations $a,\ b:\mathbb{E}_6\longrightarrow\mathbb{E}_6$

$$a=\begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \varepsilon_6 \\ \varepsilon_1 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \varepsilon_6 & \varepsilon_2 \end{pmatrix} \qquad b=\begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \varepsilon_6 \\ \varepsilon_5 & -\varepsilon_2 & -\varepsilon_3 & \varepsilon_6 & \varepsilon_1 & \varepsilon_4 \end{pmatrix}$$

where $\varepsilon_1=(1,0,0,0,0,0)$, $\varepsilon_2=(0,1,0,0,0,0)$, $\ldots$, $\varepsilon_6=(0,0,0,0,0,1)$ is the canonical basis of $\mathbb{E}_6$, defining the orthogonal representation

$$\begin{aligned} a(x_1,x_2,x_3,x_4,x_5,x_6)&=(x_1,x_6,x_2,x_3,x_4,x_5)\\ b(x_1,x_2,x_3,x_4,x_5,x_6)&=(x_5,-x_2,-x_3,x_6,x_1,x_4)\end{aligned} \tag{7}$$

of Y in $\mathbb{E}_6$. The subspace $\mathbb{E}_6^{\parallel}\subset\mathbb{E}_6$ and its orthogonal

$$\mathbb{E}_6^{\perp}=\left\{x\in\mathbb{E}_6\,\middle|\,\langle x,y\rangle=0\ \text{ for any } y\in\mathbb{E}_6^{\parallel}\right\} \tag{8}$$

are Y-invariant, and the corresponding orthogonal projectors $\pi^{\parallel},\ \pi^{\perp}:\mathbb{E}_6\longrightarrow\mathbb{E}_6$ satisfying the relations

$$\begin{aligned} &\pi^{\parallel}\circ\pi^{\parallel}=\pi^{\parallel} && \pi^{\perp}\circ\pi^{\perp}=\pi^{\perp}\\ &\pi^{\parallel}\circ\pi^{\perp}=\pi^{\perp}\circ\pi^{\parallel}=0 && \pi^{\parallel}+\pi^{\perp}=1\end{aligned} \tag{9}$$

are given by

$$\pi^{\parallel}=\mathcal{M}\left(\tfrac{1}{2},\tfrac{1}{10}\sqrt{5}\right) \qquad \pi^{\perp}=\mathcal{M}\left(\tfrac{1}{2},-\tfrac{1}{10}\sqrt{5}\right) \tag{10}$$

where

$$\mathcal{M}(\xi,\eta)=\begin{pmatrix} \xi & \eta & \eta & \eta & \eta & \eta \\ \eta & \xi & \eta & -\eta & -\eta & \eta \\ \eta & \eta & \xi & \eta & -\eta & -\eta \\ \eta & -\eta & \eta & \xi & \eta & -\eta \\ \eta & -\eta & -\eta & \eta & \xi & \eta \\ \eta & \eta & -\eta & -\eta & \eta & \xi \end{pmatrix}. \tag{11}$$

Each element $x\in\mathbb{E}_6$ can be written in the form

$$x=x^{\parallel}+x^{\perp}$$

such that $x^{\parallel}\in\mathbb{E}_6^{\parallel}$ and $x^{\perp}\in\mathbb{E}_6^{\perp}$. The elements $x^{\parallel},\ x^{\perp}$ satisfying this condition are uniquely determined and are given by $x^{\parallel}=\pi^{\parallel}x,\ \ x^{\perp}=\pi^{\perp}x$.

The vectors $v_1=\kappa e^1$, $v_2=\kappa e^2$, $v_3=\kappa e^3$, where $\kappa=1/\sqrt{2(\tau+2)}$, form an orthonormal basis of $\mathbb{E}_6^{\parallel}$, and the isometry

$$\lambda:\mathbb{E}_3\quad\longrightarrow\quad\mathbb{E}_6^{\parallel}:r\quad\mapsto\quad\left(\kappa\langle r,e_1\rangle,\ \kappa\langle r,e_2\rangle,\ \ldots\ ,\ \kappa\langle r,e_6\rangle\right) \tag{12}$$

which is an isomorphism of representations of Y with the property $\lambda(1,0,0) = v_1$, $\lambda(0,1,0) = v_2$, $\lambda(0,0,1) = v_3$, allows us to identify the physical space $\mathbb{E}_3$ with $\mathbb{E}_6^{\parallel}$. One can remark that

$$\mathbb{E}_6^{\parallel} = \left\{ \left(\langle r, e_1 \rangle, \langle r, e_2 \rangle, \ldots, \langle r, e_6 \rangle \right) \,\middle|\, r \in \mathbb{E}_3 \right\}. \tag{13}$$

The set $\mathcal{L} = c\mathbb{Z}^6 \subset \mathbb{E}_6$, where $c = 1/\kappa$, is a Y-invariant $\mathbb{Z}$-module. Since

$$\pi^{\parallel}(c\varepsilon_j) = \lambda(e_j) \tag{14}$$

for any $j \in \{1, 2, \ldots, 6\}$, in view of the considered identification

$$\pi^{\parallel}(\mathcal{L}) = \sum_{j=1}^{6} \mathbb{Z} e_j. \tag{15}$$

Let $\rho = c/2$, and let W be a set satisfying the condition

$$\pi^{\perp}\left((-\rho, \rho)^6 \right) \subset W \subset \pi^{\perp}\left([-\rho, \rho]^6 \right) \tag{16}$$

which contains one and only one point from each pair of opposite points of its boundary [9]. In particular, one can remark that

$$\|x^{\perp}\| < \rho \quad \Longrightarrow \quad x^{\perp} \in W \qquad x^{\perp} \in W \quad \Longrightarrow \quad \|x^{\perp}\| < \rho\sqrt{6}. \tag{17}$$

The vertex set corresponding to the three-dimensional Penrose tiling is [9]

$$\mathcal{Q} = \left\{ x^{\parallel} \,\middle|\, x \in \mathcal{L}, \quad x^{\perp} \in W \right\}. \tag{18}$$

If the constants $\alpha \in \mathbb{R} - \{0\}$ and $\beta \in (-1, 1]$ are such that the matrix $S = \alpha\pi^{\parallel} + \beta\pi^{\perp}$ has integer entries, then

$$\left.\begin{array}{l} x \in \mathcal{L} \\ x^{\perp} \in W \end{array}\right\} \quad \Longrightarrow \quad \left\{\begin{array}{l} Sx \in \mathcal{L} \\ (Sx)^{\perp} = \beta x^{\perp} \in W. \end{array}\right. \tag{19}$$

In this case $x^{\parallel} \in \mathcal{Q} \Longrightarrow (Sx)^{\parallel} = \alpha x^{\parallel} \in \mathcal{Q}$, that is, $r \mapsto \alpha r$ is a self-similarity of $\mathcal{Q}$.

The matrix $S = \alpha\pi^{\parallel} + \beta\pi^{\perp} = \mathcal{M}(\frac{1}{2}(\alpha+\beta), \frac{1}{10}(\alpha-\beta)\sqrt{5})$ has integer entries if and only if the numbers

$$n = \tfrac{1}{2}(\alpha + \beta) \qquad k = \tfrac{1}{10}(\alpha - \beta)\sqrt{5} \tag{20}$$

are integers. It follows that

$$\alpha = n + k\sqrt{5} \qquad \beta = n - k\sqrt{5}. \tag{21}$$

The set

$$\mathcal{C} = \left\{ n + k\sqrt{5} \,\middle|\, (n, k) \in \mathbb{Z}^2, \quad -1 < (n + k\sqrt{5})^* \leqslant 1 \right\} \tag{22}$$

where $(n + k\sqrt{5})^* = n - k\sqrt{5}$, represents a one-dimensional quasicrystal [7], and

$$\mathbb{E}_3 \quad \longrightarrow \quad \mathbb{E}_3 : r \quad \mapsto \quad \alpha r \tag{23}$$

is a self-similarity of $\mathcal{Q}$, for any $\alpha \in \mathcal{C} - \{0\}$. In particular, between the elements of $\mathcal{C}$ there is the well known scaling factor $\tau^3 = 2 + \sqrt{5}$.

For any $\alpha \in \mathcal{C} - \{0\}$ satisfying the relation $|\alpha^*| < 1/\sqrt{6}$ we denote

$$\rho_\alpha = \rho(1 - |\alpha^*|\sqrt{6})/(1 + |\alpha^*|) \qquad S_\alpha = \alpha\pi^{\parallel} + \alpha^*\pi^{\perp}. \tag{24}$$

Since $\pi^{\perp}(\mathcal{L})$ is dense in $\mathbb{E}_6^{\perp}$, it follows that the set

$$\mathcal{L}_\alpha = \left\{ y \in \mathcal{L} \,\middle|\, \|y^{\perp}\| < \rho_\alpha \right\} \tag{25}$$

is an infinite set. For each $y \in \mathcal{L}_\alpha$ and each $x \in \mathcal{L}$ such that $x^\perp \in W$ we get

$$\|(S_\alpha(x-y)+y)^\perp\| = \|\alpha^*(x^\perp - y^\perp) + y^\perp\| \leqslant |\alpha^*|\ \|x^\perp - y^\perp\| + \|y^\perp\|$$

$$\leqslant |\alpha^*|\,(\|x^\perp\| + \|y^\perp\|) + \|y^\perp\| < |\alpha^*|\sqrt{6}\rho + (|\alpha^*|+1)\rho_\alpha = \rho$$

whence $(S_\alpha(x-y)+y)^\perp \in W$. Thus

$$\left.\begin{array}{l} x \in \mathcal{L} \\ x^\perp \in W \end{array}\right\} \quad \Longrightarrow \quad \left\{\begin{array}{l} S_\alpha(x-y)+y \in \mathcal{L} \\ (S_\alpha(x-y)+y)^\perp \in W \end{array}\right. \tag{26}$$

for any $y \in \mathcal{L}_\alpha$. It follows that

$$x^\parallel \in \mathcal{Q} \quad \Longrightarrow \quad (S_\alpha(x-y)+y)^\parallel = \alpha(x^\parallel - y^\parallel) + y^\parallel \in \mathcal{Q} \tag{27}$$

for any $y \in \mathcal{L}_\alpha$, i.e. $\mathcal{Q}$ is invariant under the affine similarity

$$\mathbb{E}_3 \quad \longrightarrow \quad \mathbb{E}_3 : r \quad \mapsto \quad \alpha(r-q)+q \tag{28}$$

for any $q \in \mathcal{Q}_\alpha = \left\{ y^\parallel \,\middle|\, y \in \mathcal{L}_\alpha \right\} \subset \mathcal{Q}$.

The results obtained are similar to those recently reported by Masáková *et al* [7], but the two approaches are different. More that that, the method of Masáková *et al* does not work in the case of Penrose patterns. The only patterns of $\mathbb{E}_3$ considered in [7] are those defined by a so-called ‘star map’. Each star map of $\mathbb{E}_3$ is defined by a pair of bases $\{\alpha_1,\ \alpha_2,\ \alpha_3\}$ and $\{\alpha_1^*,\ \alpha_2^*,\ \alpha_3^*\}$ of $\mathbb{E}_3$ and has the form

$$* : M \quad \longrightarrow \quad M^* : \ \sum_{j=1}^{3}(a_j + b_j\tau)\alpha_j \quad \mapsto \quad \sum_{j=1}^{3}(a_j + b_j\tau')\alpha_j^* \tag{29}$$

where

$$M = \sum_{i=1}^{3}\mathbb{Z}[\tau]\alpha_i \qquad M^* = \sum_{i=1}^{3}\mathbb{Z}[\tau]\alpha_i^* \tag{30}$$

$\tau' = \frac{1}{2}(1-\sqrt{5})$ and $\mathbb{Z}[\tau] = \{a + b\tau \mid a, b \in \mathbb{Z}\}$. For each star map and each bounded convex and closed set $\Omega \subset \mathbb{E}_3$ with non-empty interior one obtains the quasiperiodic pattern

$$\Sigma(\Omega) = \left\{x \in M \,\middle|\, x^* \in \Omega\right\}. \tag{31}$$

The $\mathbb{Z}$-module $L = \sum_{j=1}^{6}\mathbb{Z}e_j$ does not contain the element τe_1. Indeed, the relation $\tau e_1 = \sum_{j=1}^{6} x_j e_j$ is equivalent to the system of equations

$$\begin{aligned} \tau &= x_1 + \tau x_2 + \tau x_3 - x_5 \\ 0 &= -x_2 + x_3 + \tau x_4 - \tau x_6 \\ \tau + 1 &= \tau x_1 + x_4 + \tau x_5 + x_6 \end{aligned} \tag{32}$$

which does not admit integer solutions.

Since L is not a $\mathbb{Z}[\tau]$-module ($e_1 \in L$, but $\tau e_1 \notin L$) and M is a $\mathbb{Z}[\tau]$-module, there cannot exist $\alpha_1,\ \alpha_2,\ \alpha_3$ such that $L = M$. If we choose M such that $L \subset M$, then

$$\tau e_1 + L = \left\{\tau e_1 + x \,\middle|\, x \in L\right\} \tag{33}$$

is a subset of $M - L$ and

$$(\tau e_1 + L)^* = \left\{x^* \,\middle|\, x \in \tau e_1 + L\right\} \tag{34}$$

is a dense subset of $\mathbb{E}_3$. It follows

$$\Omega \cap (\tau e_1 + L)^* \neq \emptyset \tag{35}$$

and hence, $\Sigma(\Omega) \not\subset L$.

Thus, despite the infinite number of possible choices for the bases $\{\alpha_1,\ \alpha_2,\ \alpha_3\}$, $\{\alpha_1^*,\ \alpha_2^*,\ \alpha_3^*\}$ and Ω, the method of [7] does not allow us to obtain the icosahedral pattern $\mathcal{Q}$.

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